Near Extreme Black Holes and the Universal Relaxation Bound

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A fundamental bound on the relaxation time τ of a perturbed thermodynamical system has recently been derived, $\tau \geq \hbar/\pi T$, where T is the system's temperature. We demonstrate *analytically* that black holes saturate this bound in the extremal limit and for large values of the azimuthal number m of the perturbation field.

The deep connections between the physics of black holes, and the realms of thermodynamics and information theory were revealed by Hawking's theoretical discovery of black-hole radiation [1], and its corresponding black-hole entropy and temperature [2]. These discoveries suggest that black holes behave as thermodynamic systems in many respects.

On another front, a fundamental problem in thermodynamic and statistical physics is to determine the relaxation timescale at which a perturbed physical system returns to a stationary, equilibrium configuration [3]. Recently, we have derived [4] a universal bound on the relaxation time τ of perturbed thermodynamical systems:

$$\tau_{min} = \hbar/\pi T \ , \tag{1}$$

where T is the system's temperature [5]. This bound is based on standard results from quantum information theory and thermodynamic considerations [6, 7]. Thus the relaxation time of a perturbed physical system is fundamentally bounded by the reciprocal of its temperature: the colder it gets the longer it takes to settle down to a stationary equilibrium configuration, in accord with the spirit of the third-law of thermodynamics [8].

It should be noted that mundane physical systems are characterized by relaxation times which are typically many orders of magnitude larger than $\hbar/T$ [4]. To put our relaxation bound to an interesting test, we must consider physical systems with strong self-gravity.

Back to Gravity: a stationary black hole corresponds to a thermal state, characterized by the Bekenstein-Hawking temperature [1, 2], $T_{BH} = \frac{\hbar(r_+-r_-)}{4\pi(r_+^2+a^2)}$, where $r_\pm = M + (M^2-a^2)^{1/2}$ are the black-hole outer and inner horizons, and M and a are the black-hole mass and angular momentum per unit mass, respectively. (We consider here the canonical Kerr black holes). Perturbing the black hole corresponds to perturbing this thermal state, and the decay of the perturbation (characterized by damped oscillations) describes the return to thermal equilibrium.

The response of a black hole to external perturbations is characterized by 'quasinormal ringing' (QNM), damped oscillations with a discrete frequency spectrum [9]. All perturbations of the black-hole spacetime are radiated away in a manner reminiscent of the last pure dying tones of a ringing bell: the relaxation process is characterized by an exponential decay of the form $e^{-i\omega t}$, with complex black-hole quasinormal frequencies, $\omega = \Re\omega - i\Im\omega$.

It turns out that a perturbed black hole has an infinite number of quasinormal frequencies, characterizing oscillations with decreasing relaxation times (increasing imaginary part) [10]. The mode with the smallest imaginary part (known as the fundamental mode) gives the dynamical timescale τ for generic perturbations to decay (the relaxation time required for the perturbed black hole to return to a quiescent state). Namely, $\tau \equiv \omega_I^{-1}$, where ω_I denotes the imaginary part of the fundamental, least damped black-hole resonance.

Taking cognizance of the relaxation bound Eq. (1), one deduces an upper bound on the black-hole fundamental frequency [4]

$$\omega_I \leq \pi T_{BH}/\hbar \ . \tag{2}$$

Thus the relaxation bound implies that a black hole must have (at least) one quasinormal resonance whose imaginary part conform to the upper bound (2). This mode would dominate the relaxation dynamics of the perturbed black hole, and will determine its characteristic relaxation timescale.

It has been demonstrated numerically [4] that perturbed black holes conform to the relaxation bound Eq. (2). In fact, black holes have relaxation times which are of the same order of magnitude as τ_{min}, the minimally allowed relaxation time [4]. Here we shall prove *analytically* that black holes may actually attain the fundamental relaxation bound in the extremal limit.

Mashhoon [11] has calculated the QNM spectrum of rotating black holes in the eikonal limit $l = |m| \gg 1$, where m is the azimuthal number of the perturbation field. The Kerr QNM frequencies in the $l = m \gg 1$ limit are given by [11]

$$\omega_n = m\omega_+ - i(n+\frac{1}{2})\beta\omega_+ \quad ; \quad n = 0, 1, 2, \dots \ , \tag{3}$$

where

$$\omega_+ \equiv \frac{M^{1/2}}{r_{ph}^{3/2}+aM^{1/2}}\ , \tag{4}$$

is the Kepler frequency for null rays in the unstable equatorial circular orbit of the black hole, and

$$r_{ph} \equiv 2M\{1+\cos[\frac{2}{3}\cos^{-1}(-a/M)]\}\ , \tag{5}$$

is the limiting circular photon orbit. The function β is given by [11]

$$\beta = \frac{(12M)^{1/2}(r_{ph}-r_+)(r_{ph}-r_-)}{r_{ph}^{3/2}(r_{ph}-M)} \tag{6}$$

We now focus on the extremal limit, $T_{BH} \to 0$. Let $r_\pm = M \pm \epsilon$, where $\epsilon \ll 1$ [this implies $T_{BH} = \epsilon/4\pi M^2 + O(\epsilon^2)$]. After some algebra we find that Eq. (5) yields

$$r_{ph} = M + \frac{2\epsilon}{\sqrt{3}} + O(\epsilon^2)\ , \tag{7}$$

which in turn implies

$$\omega_+ = \frac{1}{2M} - \frac{\sqrt{3}\epsilon}{4M^2} + O(\epsilon^2)\ , \tag{8}$$

and

$$\beta = \frac{\epsilon}{M} + O(\epsilon^2). \tag{9}$$

Substituting Eqs. (8) and (9) in Eq. (3), we find that the near-extreme black-hole quasinormal resonances are given by

$$\hbar\omega_n = m\hbar\Omega - i2\pi T_{BH}\cdot(n+\frac{1}{2})\ , \tag{10}$$

where $\Omega \equiv \frac{a}{r_+^2+a^2} = 1/2M + O(\epsilon)$ is the angular velocity of the black-hole event horizon.

It has long been known qualitatively that the QNM tend to cluster near the real axis in the extreme black-hole limit [12]. However, the quantitative description of the imaginary part which is the focus of the present analysis [that is, $\Im(\omega_n) = 2\pi T_{BH}(n+\frac{1}{2})$ in the $T_{BH} \to 0$ limit] is a new result and was not established explicitly in previous studies.

Finally, taking $n = 0$ in Eq. (10), one finds

$$\frac{\hbar\omega_I}{\pi T_{BH}} \to 1^-\ , \tag{11}$$

for the fundamental, least damped black-hole resonance [13]. This implies that Kerr black holes have the remarkable property of saturating the universal relaxation bound in the extremal limit. In this limit, the black hole has the shortest possible relaxation time for a given temperature.

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